Radio Astronomy, Spring 2024

Problem Set III

Deadline: 5PM of Monday, November 18, 2024

1. **Kirchhoff's law and Einstein coefficients (10%).** The condition of LTE renders Kirchhoff's law,
$$\frac{\varepsilon_\nu}{\alpha_\nu} = B_\nu(T),$$
where ε_ν is the emissivity, α_ν the absorption coefficient, and $B_\nu(T)$ the Planck function.

   (a) (5%) Estimate the ratio of the emissivity to the absorption coefficient for CO $(1-0)$ at $\nu = 115.271$ GHz and $T = 20$ K, typical for the millimeter waveband. Repeat for $\nu = 4.8$ GHz and $T = 10$ K, typical for the centimeter waveband.

   (b) (5%) Consider a two-level system with equal statistical weights and nearly equal level populations N_1 and N_2. In the limit $h\nu \ll kT$, use Kirchhoff's law to find the ratio of the Einstein coefficients A/B, from $\varepsilon_\nu = h\nu A N_2/4\pi$ and $\alpha_\nu = (h\nu)^2 N_1 B/kcT$. Determine the ratio of the Einstein A to B coefficients for the values in part (a) and thus determine the relative influence of stimulated and spontaneous emission as a function of T.

2. **Two-level systems (20%)**. Let's investigate the variation of excitation temperature, $T_{\rm ex}$, with the collision rate, C_{21} ad the spontaneous emission rate, A_{21}, for a two-level system. In class, we have shown that the excitation temperature can be expressed as
$$T_{\rm ex} = T_k \left(\frac{T_0 C_{21} + T_b A_{21}}{T_0 C_{21} + T_k A_{21}} \right),$$
where T_k is the kinetic temperature, T_b the brightness temperature of the radiation field, and $T_0 \equiv h\nu/k$.

   (a) (5%) The collision rate is given by $C_{21} = n_{\rm H_2} K_{21} = n_{\rm H_2} \langle \sigma v \rangle$, where the collision rate coefficient is $K_{21} \equiv \langle \sigma v \rangle \sim 10^{-10}$ cm^3 s^{-1}. When $C_{21} = A_{21}$ for the transition involved, the is referred to as the critical density, $n_{\rm crit}$. For the atomic hydrogen 21 cm line, $A_{21} = 2.85 \times 10^{-15}$ s^{-1}. Find $n_{\rm crit}$ for this transition.

   (b) (5%) For multi-level systems, computation of the level population becomes more sophisticated. However, for the purpose of comparison, repeat the calculation for the HCO^+ $(1-0)$ at $\nu = 89.188$ GHz with $A_{21} = 3 \times 10^{-5}\ \mathrm{s}^{-1}$. Compare the value of n_{crit} with the one from part (a).

(c) (5%) Now, take $T_k = 100$ K, typical for hot molecular cores, and assume a radiation field given by the cosmic microwave background at $T_b = 2.73$ K. Solve for the local density, $n_{\mathrm{H_2}}$, and evaluate its value for the HCO^+ $(1-0)$ emission with $T_{\mathrm{ex}} = 3.5$ K.

(d) (5%) For the same density in part (c), compute T_{ex} for the CO $(1-0)$ transition with $A_{21} = 7.4 \times 10^{-8}\ \mathrm{s}^{-1}$. How does this value compare to the kinetic temperature T_k?