Chapter 3

Heat Transfer in Stars

Heat transfer by random motion
Heat transfer by convection
Temperature gradients in stars
Cooling of white dwarfs

Basic Mechanisms

- **Heat transfer by random motion**
  - Thermal conduction: random motion of mass particles
  - Radiative diffusion: random motion of photons
- **Heat transfer by convection**
  - Collective motion of mass particles
  - Transferring heat by rising pockets of hot buoyant fluid and by falling pockets of cool dense fluid

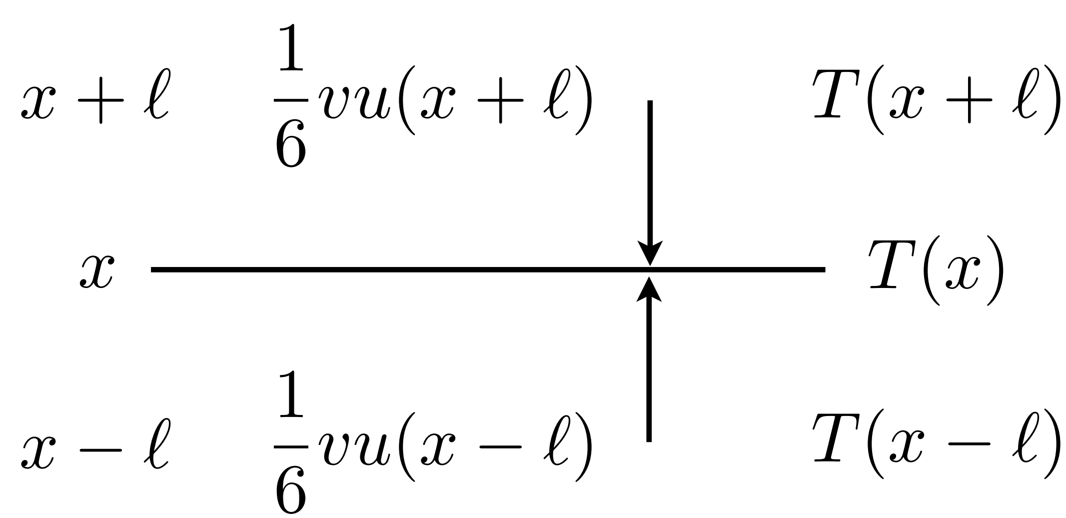
- **Criterion for the onset of convection**

Thermal Conduction I

The microscopic mechanism underlying a heat flow is the random motion of the gas particles. Assume that 1/6 of the particles move in the x-direction with a speed v and the particles travel a distance ℓ before they interact. The thermal energy density is denoted by $u(x)$. If there is a temperature gradient, the particles which cross the surface from below will have a different $u(x)$ from those which cross the surface from above.

As a result, there is a net transfer of energy across the surface, and the rate of energy transfer per unit area of the surface is

$$
\begin{aligned}
j(x) &\approx \frac{1}{6} v u(x-\ell) - \frac{1}{6} v u(x+\ell) \\
&\simeq -\frac{1}{3} v \ell \frac{\mathrm{d}u}{\mathrm{d}x}.
\end{aligned}
$$



Thermal Conduction II

Substituting the heat capacity, C_V, we have

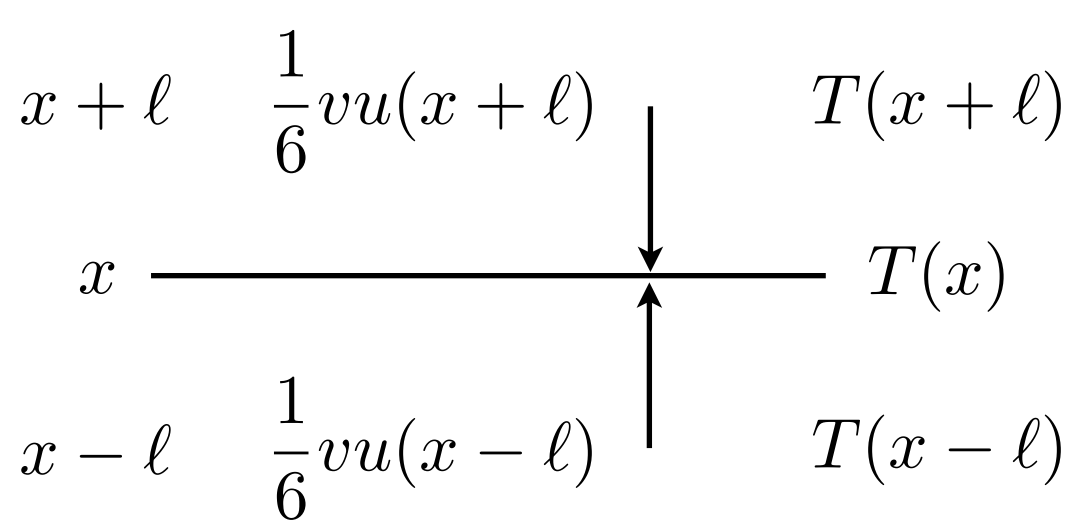
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}x} = C_V \frac{\mathrm{d}T}{\mathrm{d}x}.$$

The flux density of heat across the surface is directly proportional to the temperature gradient and the heat capacity as

$$\boxed{j(x) = -K\frac{\mathrm{d}T}{\mathrm{d}x} \quad \text{and} \quad K \approx \frac{1}{3} v\ell C_V,}$$

where K is the **coefficient of thermal conductivity** of the gas.

When the particles have a distribution of speeds, a more sophisticated calculation gives a similar result for K but with v and l replaced by the mean speed $\bar{v}$ and the mean free path $\bar{\ell}$.



Thermal Conduction III

Thermal conduction by electrons. For classical e$^-$'s with density n_e at temperature T,

$$u_e = \frac{3}{2} n_e kT, \quad C_{V,e} = \frac{3}{2} n_e k, \quad \text{and} \quad \bar{v}_e \approx \left(\frac{3kT}{m_e}\right)^{1/2}.$$

Because e^--ion collisions are more effective at transferring energy than e^--e^- collisions, the mean free path, ℓ, should concern the mean free path for an e^- to collide with an ion. Note that

$$\ell = \frac{1}{n_i\,\sigma_{ie}},$$

where n_i is the density of ions and σ_{ie} is the e^--ion collision cross-section. Consequently, $\sigma_{ie} = \pi r_{ie}^2$, where r_{ie} is the distance at which the potential energy of an e^--ion pair is comparable to the thermal kinetic energy

$$\frac{Ze^2}{r_{ie}} \approx kT.$$

Thermal Conduction IV

The coefficient of thermal conductivity due to electrons is given by

$$K_e \approx \frac{k}{2\pi}\frac{n_e}{n_i}\left[\frac{3kT}{m_e}\right]^{1/2}\left[\frac{kT}{Ze^2}\right]^2 .$$

Thermal conduction by ions. Similarly, we can obtain the thermal conductivity due to ions, K_i, by interchanging n_e and n_i as well as m_e and m_i. Assuming a fully ionized plasma with $n_e = Zn_i$, we find

$$K_i = \frac{1}{Z^2}\left[\frac{m_e}{m_i}\right]^{1/2} K_e .$$

Since $Z > 1$ and $m_i \gg m_e$, it follows that $K_i \ll K_e$, a result reflecting the fact that ions are outnumbered by e^-'s and that ions move less quickly than e^-'s. The random thermal motion of ions is a less effective mechanism for heat transfer than the random thermal motion of e^-'s. In fact, thermal conductivity by e^-'s and ions is of minor importance in most stars except in white dwarfs, where e^-'s form a dense, degenerate gas with very high thermal conductivity.

Radiative Diffusion I

Thermal conduction of heat by photons is referred to as **radiative diffusion**. Thermal photons form a gas with an energy density and a heat capacity

$$u_r = aT^4 \quad \text{and} \quad C_{V,r} = 4aT^3.$$

The heat flux density due to radiative diffusion is then

$$j(x) = -K_r \frac{\mathrm{d}T}{\mathrm{d}x} \quad \text{with} \quad K_r \approx \frac{4}{3} c \bar{\ell} a T^3,$$

where K_r can be considered as the coefficient of thermal conduction due to the random motion of photons.

Let's estimate the mean free path for a photon collision in the simplest case with high temperatures and low densities found in the interiors of massive main sequence stars. The dominant process is Thomson scattering by e^-'s and

$$\bar{\ell} = \frac{1}{n_e \sigma_T}, \quad \text{where } \sigma_T = \frac{8\pi}{3} \left[\frac{e^2}{m_e c^2} \right]^2.$$

Radiative Diffusion II

Thermal conduction by photons. The coefficient for thermal conduction by photons is thus

$$K_r \simeq \frac{c}{2\pi} \frac{aT^3}{n_e} \left[\frac{m_e c^2}{e^2} \right]^2 .$$

When comparing to the coefficient for thermal conduction by electrons, we find

$$\frac{K_r}{K_e} \approx \sqrt{3} Z \frac{P_r}{P_e} \left[\frac{m_e c^2}{kT} \right]^{5/2} ,$$

where $P_r = aT^4/3$ and $P_e = n_e kT$ are the radiation and electron pressures.

Consider the solar interior, as an example, with $T = 6 \times 10^6$ K and $\rho = 1.4\ \mathrm{g\,cm^{-3}}$, which gives $P_r = 3 \times 10^{12}\ \mathrm{dyn\,cm^{-2}}$ and $P_e = 7 \times 10^{14}\ \mathrm{dyn\,cm^{-2}}$. Thus, $K_r \approx 2 \times 10^5\, K_e$ and concludes that radiative diffusion is a more effective mechanism for heat transfer in the sun than thermal conduction by e^-'s.

Thomson Scattering I

Thomson scattering (electron scattering) is the process in which a free charge, e.g. e^-, radiates in response to an incident electromagnetic wave, i.e. a photon. The force due to a linearly polarized wave is

$$\boldsymbol{F} = e\boldsymbol{\varepsilon} E_0 \sin \omega_0 t,$$

where $\boldsymbol{\varepsilon}$ is the unit vector of the $\boldsymbol{E}$-field. The response of the free e^- to this force is described by

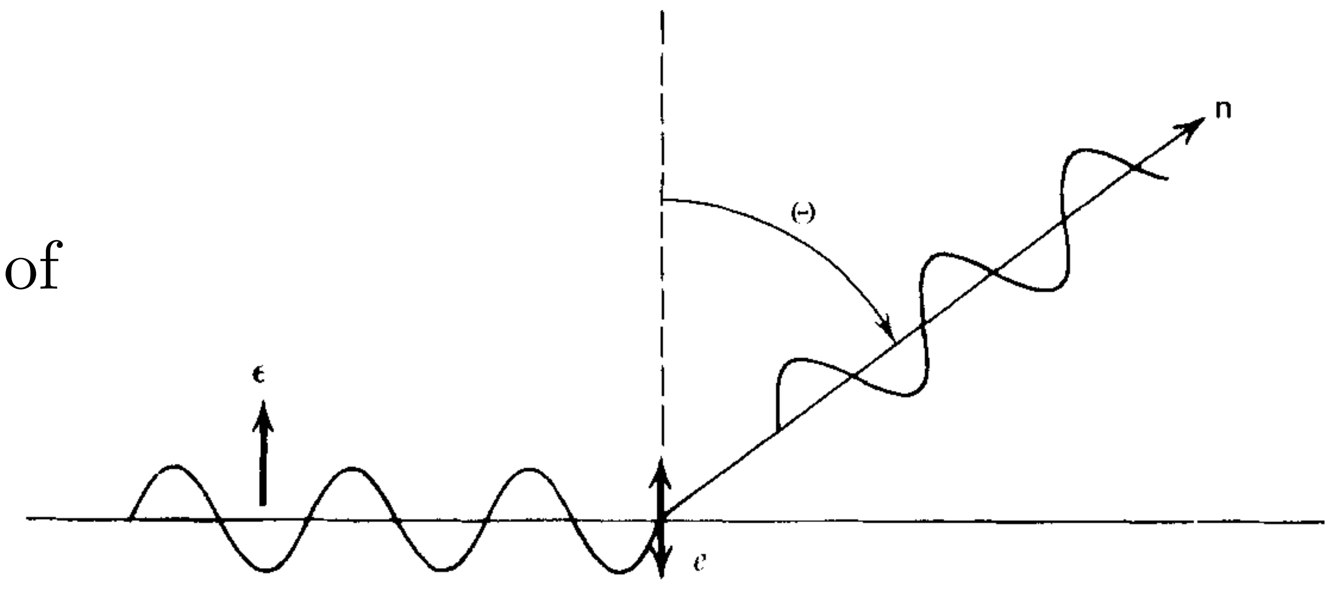
$$m\ddot{\boldsymbol{r}} = e\boldsymbol{\varepsilon} E_0 \sin \omega_0 t.$$

Rewriting with the dipole moment, $\boldsymbol{d} = e\boldsymbol{r}$, we have

$$\ddot{\boldsymbol{d}} = \frac{e^2 E_0}{m} \boldsymbol{\varepsilon} \sin \omega_0 t$$
$$\boldsymbol{d} = -\left(\frac{e^2 E_0}{m_e \omega_0^2}\right) \boldsymbol{\varepsilon} \sin \omega_0 t,$$

which describes an oscillating dipole of amplitude

$$\boldsymbol{d_0} = \frac{e^2 E_0}{m \omega_0^2} \boldsymbol{\varepsilon}.$$



Thomson Scattering II

Using the dipole approximation, we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\ddot{\boldsymbol{d}}^2}{4\pi c^3}\sin^2\Theta = \frac{e^4 E_0^2}{8\pi\, m_e^2\, c^3}\sin^2\Theta.$$

Comparing with the incident flux of $\langle S\rangle = \frac{c}{8\pi}E_0^2$, one can define the differential cross-section, $\mathrm{d}\sigma$ for scattering into $\mathrm{d}\Omega$ and obtain

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \langle S\rangle\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{cE_0^2}{8\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

$$\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left[\frac{e^2}{m_e\, c^2}\right]^2\sin^2\Theta,$$

where $r_e \equiv e^2/m_e c^2 = 2.82\times 10^{-13}$ cm is called the **classical electron radius**. The total cross-section of Thomas scattering can be found by integrating over solid angle, using $\mu \equiv \cos\Theta$,

$$\int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = 2\pi r_e^2\int_{-1}^{1}(1-\mu^2)\mathrm{d}\mu \equiv \boxed{\sigma_T = \frac{8\pi}{3}r_e^2.}$$

Please refer to Sec. 3.4 in Rybicki & Lightman for more details.

Opacity I

Photon absorption normally involves an interaction with an e^- in the presence of an ion and becomes increasingly important at higher density and lower temperature. If the e^- is initially bound to the ion, we have bound-free absorption (a.k.a. photo-ionization), and if the e^- is initially unbound, we have free-free absorption (a.k.a. inverse-bremsstrahlung). Both bound-free and free-free absorption lead to a mean free path varying with the frequency, $\bar{\ell}_\nu$. Given the Planck function, the energy density and the heat capacity are

$$u_\nu \mathrm{d}\nu = \frac{h\nu}{e^{h\nu/kT} - 1} 8\pi \frac{\nu^2}{c^3} \mathrm{d}\nu$$
$$C_{V,\nu} \mathrm{d}\nu = \frac{\partial u_\nu}{\partial T} \mathrm{d}\nu$$

If $\bar{\ell}_\nu$ is the mean free path at frequency ν, the coefficient of conduction due to photons of all frequencies is

$$K_r = \int_0^\infty \frac{1}{3} c \bar{\ell}_\nu C_{V,\nu} \mathrm{d}\nu \approx \frac{4}{3} c \bar{\ell} a T^3,$$

where the mean free path averaged over frequency $\bar{\ell}$ is introduced.

Opacity II

The mean free path averaged over frequency is referred to as the **Rosseland average** and is defined as

$$\boxed{\bar{\ell} = \frac{\int_0^\infty \bar{\ell}_\nu C_\nu \mathrm{d}\nu}{4aT^3}.}$$

The Rosseland average is likely to be dominated by contributions at frequencies near $2.8kT/h$ where $C_{V,\nu}$ is a maximum and at frequencies where $\bar{\ell}_\nu$ is large, namely, where the stellar interior is almost transparent.

In stellar interiors, the photon mean free path is determined by the probability of an interaction with either an e^- or an ion. Given a medium with density of e^-'s and ions, n_e and n_i, and interaction cross-section, σ_e and σ_i, the probability of interaction in a distance $\mathrm{d}x$ is $(n_e\sigma_e + n_i\sigma_i)\mathrm{d}x$, and the mean free path is

$$\bar{\ell} = \frac{1}{n_e\sigma_e + n_i\sigma_i}.$$

Opacity III

In practice, the **opacity** is introduced to describe the mean free path in a medium of mass density ρ

$$\boxed{\bar{\ell} = \frac{1}{\rho\kappa}.}$$

The flux density of heat conduction by radiation is given by

$$\boxed{j(x) = -\frac{4ac}{3}\frac{T^3}{\rho\kappa}\frac{\mathrm{d}T}{\mathrm{d}x}.}$$

Kramers' law. Bound-free absorption is important at low temperatures where a large fraction of atoms are partially ionized while free-free absorption dominates at higher temperatures where ionization nears completion. Over all, the Rosseland mean opacity increases with density and decreases with temperature as

$$\boxed{\kappa \propto \rho T^{-3.5}.}$$

Opacity IV

At $T \sim 10^4$ K, neutral H and He start to become ionized.
At $T \sim 4 \times 10^4$ K, He II is further ionized.
Above $T \sim 10^5$ K, the ionization of certain metals, most notably Fe, occurs.
At the highest temperatures, nearly all the stellar material is ionized and electron scattering dominates to produce a constant opacity

$$\kappa_{es} = \frac{n_e \sigma_T}{\rho} = (1+X)\frac{\sigma_T}{2m_{\rm H}} \approx (1+X)\,0.2\ {\rm cm^2\,g^{-1}}.$$

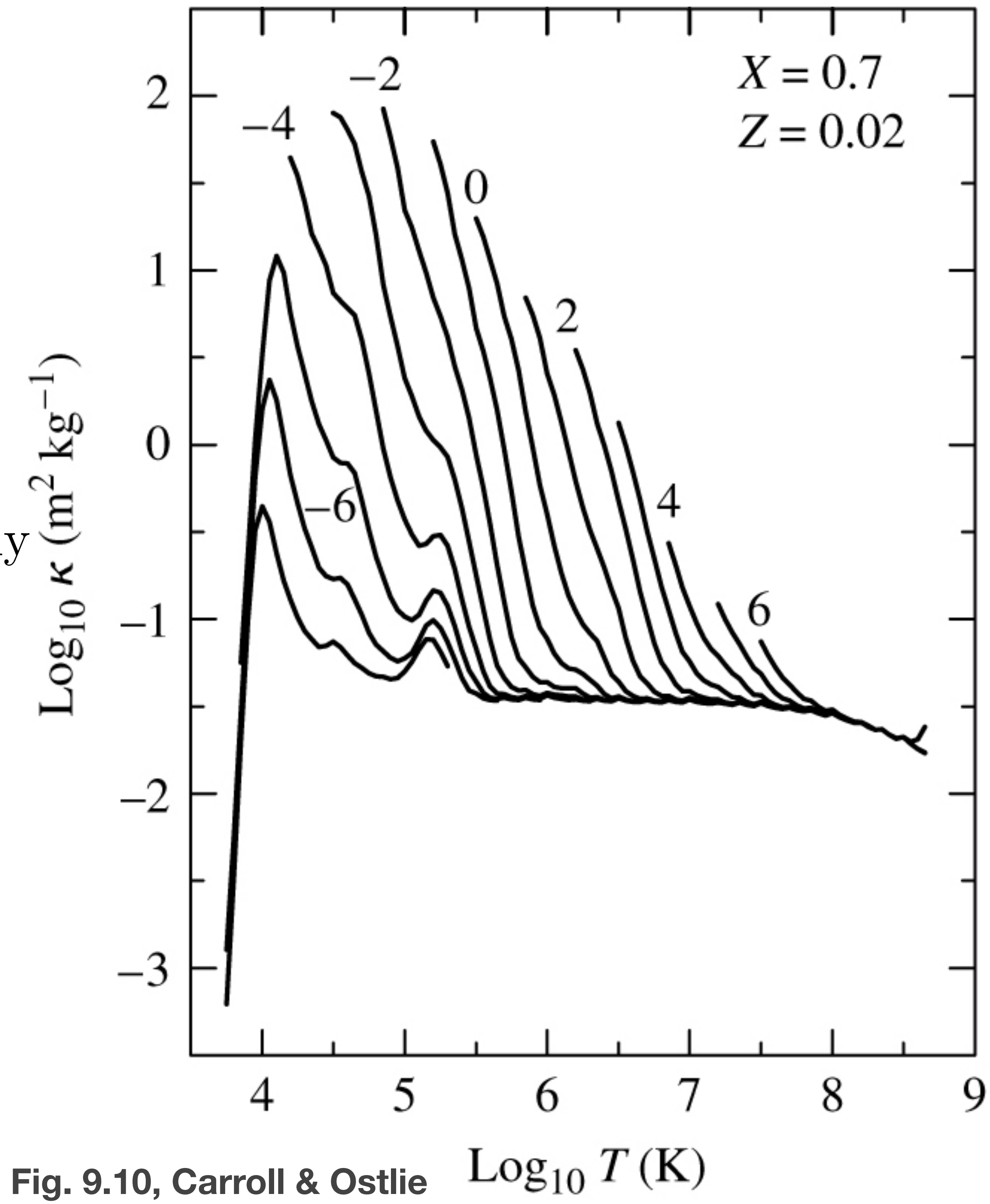


Fig. 9.10, Carroll & Ostlie

Example of the Solar Interior

Take the Sun as an example. The opacity of solar material at a density of 10 $\mathrm{g\,cm^{-3}}$ and a temperature of 2×10^6 K, is about 10^2 $\mathrm{cm^2\,g^{-1}}$, corresponding to a photon mean free path of

$$\bar{\ell} = \frac{1}{\rho\,\kappa} = 10^{-3}\ \mathrm{cm}.$$

At a higher temperature of 10^7 K, the opacity is smaller and the mean free path longer, roughly 1 $\mathrm{cm^2\,g^{-1}}$ and 0.1 cm.

Convection I

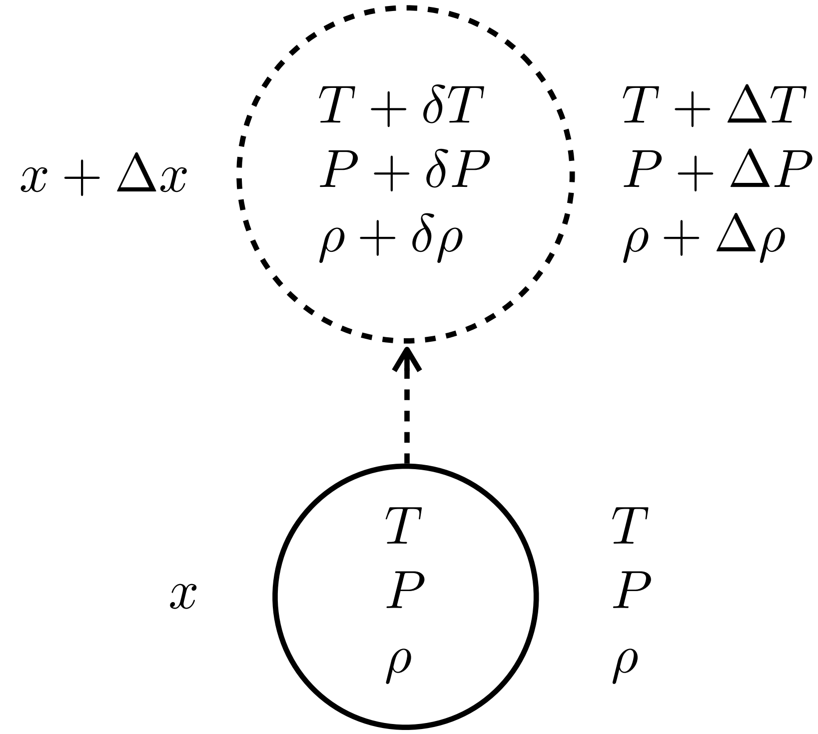
In the presence of a force field such as gravity, heat may be transferred by the collective motion of the particles. Because of buoyancy, a rising pocket of stellar gas may continue to rise when it find itself in a cooler, more dense environment. Conversely, a falling pocket of gas will continue to fall if it finds itself in a warmer, less dense environment.

Criterion for convection. Consider a pocket of gas at height x and later displaced to a height $x + \Delta x$. Assume that the pressure inside the pocket responds rapidly to the new environment so that

$$\delta P = \Delta P.$$

We also assume that there is insufficient time for heat conduction to the environment so the displaced pocket of gas *expanded adiabatically* until its pressure matches the surrounding pressure. For an adiabatic process $P \propto \rho^\gamma$, so that

$$\frac{\delta \rho}{\rho} = \frac{1}{\gamma} \frac{\delta P}{P}.$$



Convection II

For the convection to occur, the pocket needs to be buoyant, and will continue to rise, if it contains gas which is less dense than the surrounding gas

$$\delta\rho < \Delta\rho, \quad \text{or} \quad \frac{1}{\gamma}\frac{\delta P}{P} < \frac{\Delta P}{P} - \frac{\Delta T}{T}.$$

Give $\delta P = \Delta P$, the condition for convection can be rewritten as

$$\frac{\Delta T}{T} < \frac{\gamma - 1}{\gamma}\frac{\Delta P}{P}$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}\ln P}{\mathrm{d}\ln T} < \frac{\gamma}{\gamma - 1}.}$$

In other words, the critical temperature gradient for the onset of convection is given by

$$\frac{\mathrm{d}T}{\mathrm{d}x} < \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}x}.$$

Note that the negative nature of both temperature and pressure gradients in stellar interiors, i.e. $\mathrm{d}T/\mathrm{d}x < 0$ and $\mathrm{d}P/\mathrm{d}x < 0$.

Convection III

The **adiabatic index**, γ, is related to the number of classical degrees of freedom of the particles. If there is s classical degrees of freedom, each with an average thermal energy of $\frac{1}{2}kT$, we have

$$\gamma = \frac{C_P}{C_V} = \frac{1 + s/2}{s/2}.$$

For gas particles with just three translational degrees of freedom, $s = 3$ and $\gamma = 5/3$. Note that $\gamma \to 1$ when s becomes large. The critical temperature gradient for convection becomes less steep and convection can start. This occurs when the gas particles can absorb heat by exciting internal degrees of freedom such as rotation, vibration, or even phase transition, e.g. ionization.

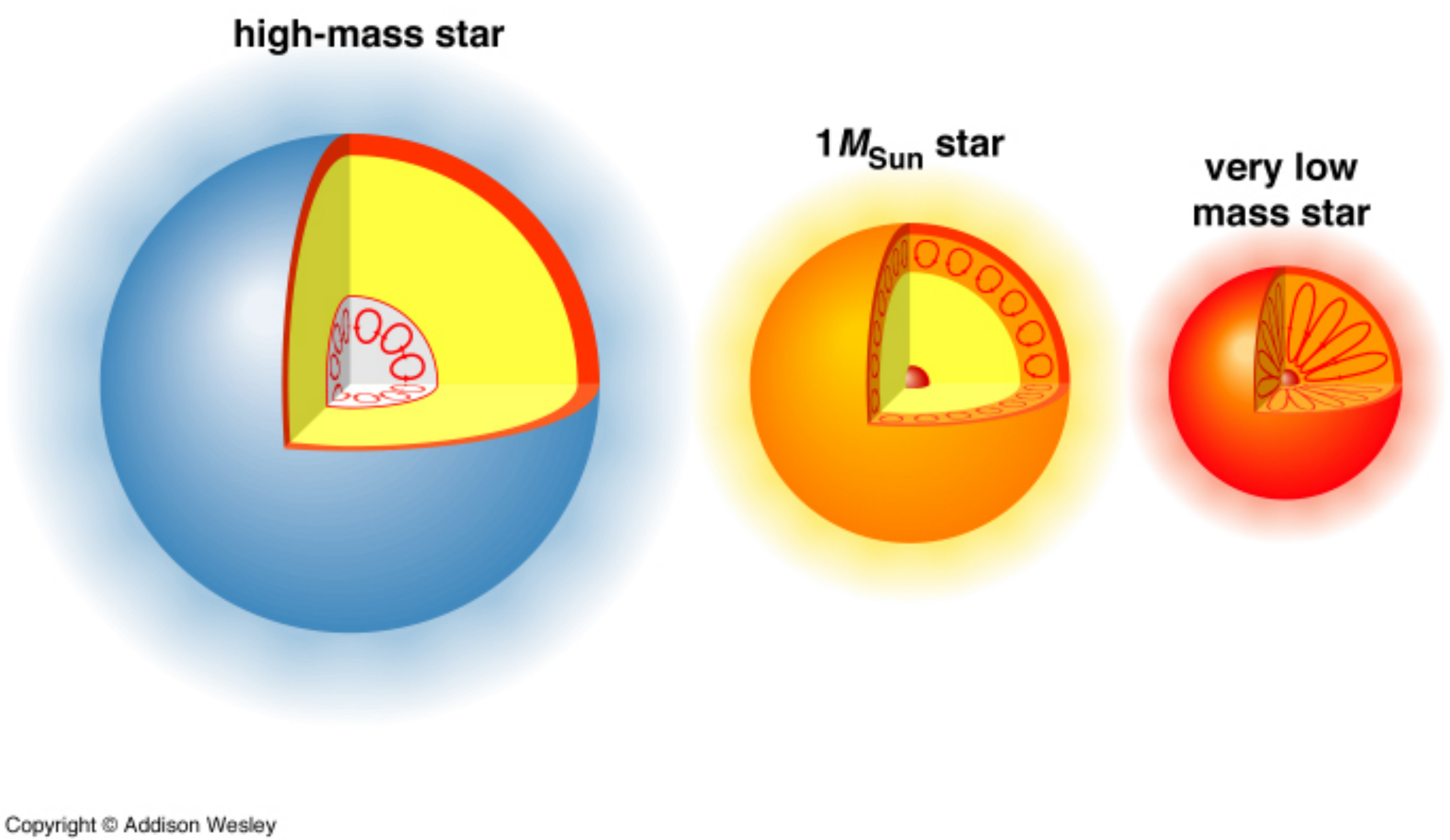
Assuming hydrostatic equilibrium in a gravitational force field, we have

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -g\,\rho(x).$$

Note that in regions where g is small, the pressure falls off gradually and convection is more easily induced.

The Interiors of Stars

- Stars of different mass have different interior structures



T Gradient in Stars I

Let $L(r)$ denote the rate at which the energy flows outwards through a spherical surface of radius r within a star. If $\varepsilon(r)$ denotes the nuclear power generated per unit volume at r, we have

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \varepsilon(r).$$

Outside the central generating regions, $L(r)$ becomes constant.

Temperature gradient for photon diffusion. Assuming that radiative diffusion is the dominant heat transfer mechanism, the total outward power flow is $L(r) = 4\pi r^2 j(r)$ and then

$$\frac{L(r)}{4\pi r^2} = -\frac{4ac\,T(r)^3}{3\rho(r)\kappa(r)}\frac{\mathrm{d}T}{\mathrm{d}r}.$$

It is more useful to consider how a star manages to transport the power generated in the interior towards the surface. If it does so by radiative diffusion, the star sets up a temperature gradient

$$\boxed{\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} = -\frac{3\rho(r)\kappa(r)}{4ac\,T(r)^3}\frac{L(r)}{4\pi r^2}.}$$

T Gradient in Stars II

Example: For the solar interior, the power flow in the Sun reaches a constant value of $L = 4\times10^{33}$ erg s^{-1} at $r = 0.4\ R_\odot$, where $T \approx 5\times10^6$ K, $\rho \approx 5\ \mathrm{g\,cm^{-3}}$, and $\kappa \approx 5\ \mathrm{cm^2\,g^{-1}}$. We obtain a temperature gradient of about $-3\ \mathrm{K\,cm^{-1}}$. Note that the fractional change in temperature over a distance comparable with the photon mean free path, in this case is 0.4 mm, is only 2×10^{-12}. Thus, radiative diffusion is valid in the solar interior, which is dense and opaque.

Temperature gradient for convection. In the interior of massive stars, the temperature gradient can reach the critical value for the onset of convection

$$\left.\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{conv}} = \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r},$$

where the pressure gradient is determined by hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}.$$

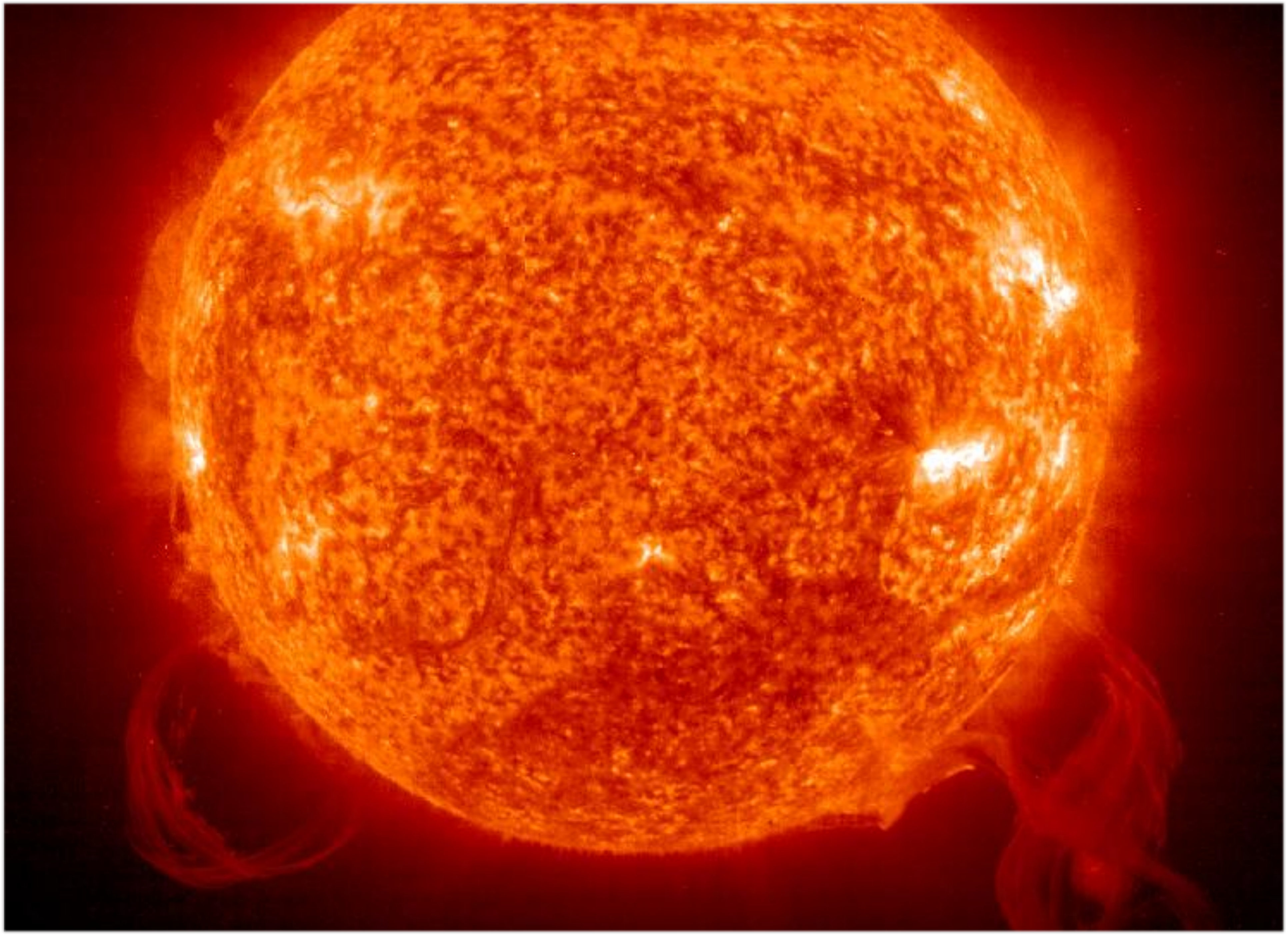
In practice, convection dominates radiative diffusion whenever the temperature gradient reaches the adiabatic critical value.

T Gradient in Stars III

Example: Ionization zones occur in the surface layers of some stars, where atoms and ions are continuously absorbing and releasing energy by ionization and recombination. The opacity, κ, is large and the temperature gradient for radiative diffusion is steep. Meanwhile, the temperature gradient for the onset of convection is not steep and encourages convection because $\gamma \to 1$.

The Sun also has a convection zone about 0.1 to $0.2\,R_\odot$ just below the photosphere. This layer appears at the base of the photosphere as bright, irregular, and transient formations, called granules. The convected energy is dissipated in the photosphere and transferred to the solar atmosphere by radiative diffusion.

Convection is also important in core of the massive stars where thermonuclear power is generated in a small core region via the CNO cycle. Large amounts of energy flow through a region where the gravitational acceleration is low and the pressure falls off gradually.



The Sun

T Gradient in Stars IV

Convective cores in massive stars. Whether a core is convective or not can be determined by setting the radiative temperature gradient equal to the critical gradient for convection

$$\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{Gm(r)\rho}{r^2}$$

$$\Rightarrow \boxed{\left.\frac{L(r)}{m(r)}\right|_{\mathrm{crit}} = \frac{\gamma - 1}{\gamma}\frac{16\pi Gc}{\kappa}\frac{P_r}{P}.}$$

Thus, a convective core occurs when

$$\boxed{\left.\frac{L(r)}{m(r)}\right|_{\mathrm{core}} > \left.\frac{L(r)}{m(r)}\right|_{\mathrm{crit}}.}$$

Convection occurs in cores of massive main sequence stars, where H-burning takes place by the CNO cycle and depends strongly on temperature to T^{20}. Such sensitive temperature dependence confines the thermonuclear fusion in a small region in which convection dominates. For less massive cores, the H-burning is via the *p*-*p* chain which is less temperature sensitive and generates energy in a larger region.

Cooling of White Dwarfs I

A white dwarf (WD) is composed of a dense system of classical ions and degenerate e^-'s, surrounded by a thin envelope of classical gas. The WD cools predominantly by the conduction of heat by e^-'s in the interior, and by the diffusion of radiation through the outer envelope. The cooling time is very long, about 10^9 yr due to the high thermal energy of the ions in the interior and the high opacity of the gas in the envelope.

If $\epsilon_F \gtrsim kT$, the degenerate e^- gas has the typical speed increased by $\sqrt{\epsilon_F/kT}$, the thermal capacity reduced by about kT/ϵ_F (however, you do not have a shortcut to derive this). The e^- mean free path, ℓ, is longer because an e^- can only be scattered if there is an unoccupied state available to be filled. The thermal conductivity of degenerate e^-'s is large that the interior of a WD is in uniform temperature, T_I.

Toy model. Consider a cooling WD consisting of a hot, metal-like, sphere surrounded by an insulating jacket of ionized gas. The thermal energy of ions, $\frac{3}{2}kT_I$, is lost to the outer envelope by photon diffusion. As the energy is lost, there is little change in the structure of a WD because it is supported by degenerate e^-'s which cannot lose energy.

Cooling of White Dwarfs II

We would like to describe the cooling of a WD, i.e. the luminosity, L, as a function of time, by rewriting physical variables in terms of the interior temperature, T_I. As a first step, let us consider the variation of P, T, and ρ in the outer envelope of the WD. Assume that 90% of the mass in the outer envelope is He and 10% is metals and that the opacity is dominated by bound-free absorption so that

$$\kappa = \kappa_0 \rho T^{-3.5} = 4.34 \times 10^{20} \rho T^{-3.5} \text{ cm}^2 \text{ g}^{-1}.$$

Applying the idea gas law, $P = \rho k T / \bar{m}$, we find

$$\kappa = \left[\frac{\kappa_0 \bar{m}}{k}\right] P T^{-4.5}.$$

Given a pressure gradient determined by hydrostatic equilibrium and a temperature gradient produced by photon diffusion

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho(r)}{r^2} \quad \text{and} \quad \frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\rho(r)\kappa(r)}{4acT(r)^3}\frac{L}{4\pi r^2},$$

we find

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \left[\frac{16\pi ac\,G}{3}\frac{M}{L}\right]\frac{T^3}{\kappa}.$$

Cooling of White Dwarfs III

Substitution of the opacity will produce the differential equation relating P and T in the envelope

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \left[\frac{16\pi ac\,Gk}{3\kappa_0\,\bar{m}}\frac{M}{L}\right]\frac{T^{7.5}}{P}.$$

Integrating with the boundary condition that $P = 0$ when $T = 0$, we find that

$$\frac{P^2}{2} = \left[\frac{16\pi ac\,Gk}{3\kappa_0\,\bar{m}}\frac{M}{L}\right]\frac{T^{8.5}}{8.5}.$$

As one goes deeper into the WD, P, T, and ρ increase until the e^-'s become degenerate at some point where $n_e \gg n_{\rm QNR}$. Given the fact that 2/3 of the particles in a WD are e^-'s, we have

$$n_e = \frac{2}{3}\frac{P}{kT} = \frac{2}{3k}\left[\frac{1}{4.25}\frac{16\pi ac\,Gk}{3\kappa_0\,\bar{m}}\frac{M}{L}\right]^{1/2} T^{13/4}.$$

Make an approximation that $n_e = 10 n_{\rm QNR}$ and obtain the interior temperature

$$\frac{2}{3k}\left[\frac{1}{4.25}\frac{16\pi ac\,Gk}{3\kappa_0\,\bar{m}}\frac{M}{L}\right]^{1/2} T_I^{13/4} = 10\left[\frac{2\pi m_e k T_I}{h^2}\right]^{3/2}.$$

Cooling of White Dwarfs IV

Use the Sun as a standard, we find the following estimate

$$T_I \approx 7 \times 10^7 \text{ K} \left[\frac{L/L_\odot}{M/M_\odot} \right]^{2/7} .$$

In other words, the luminosity of a WD can be expressed in terms of its mass and isothermal interior temperature

$$L \approx \left[\frac{T_I}{7 \times 10^7 \text{ K}} \right]^{7/2} \left[\frac{M}{M_\odot} \right] L_\odot .$$

The energy source of a white dwarf is the thermal energy of the classical ions in the interior given by

$$E \approx \frac{3}{2} N k T_I = \frac{3}{2} \left[\frac{M}{12 m_{\rm H}} \right] k T_I .$$

If the WD was formed following the completion of He-burning, $T_I \approx 10^8$ K and the initial luminosity is $1 L_\odot$ if the mass is $0.4 M_\odot$. The thermal energy stored in the ions is about 8×10^{47} erg.

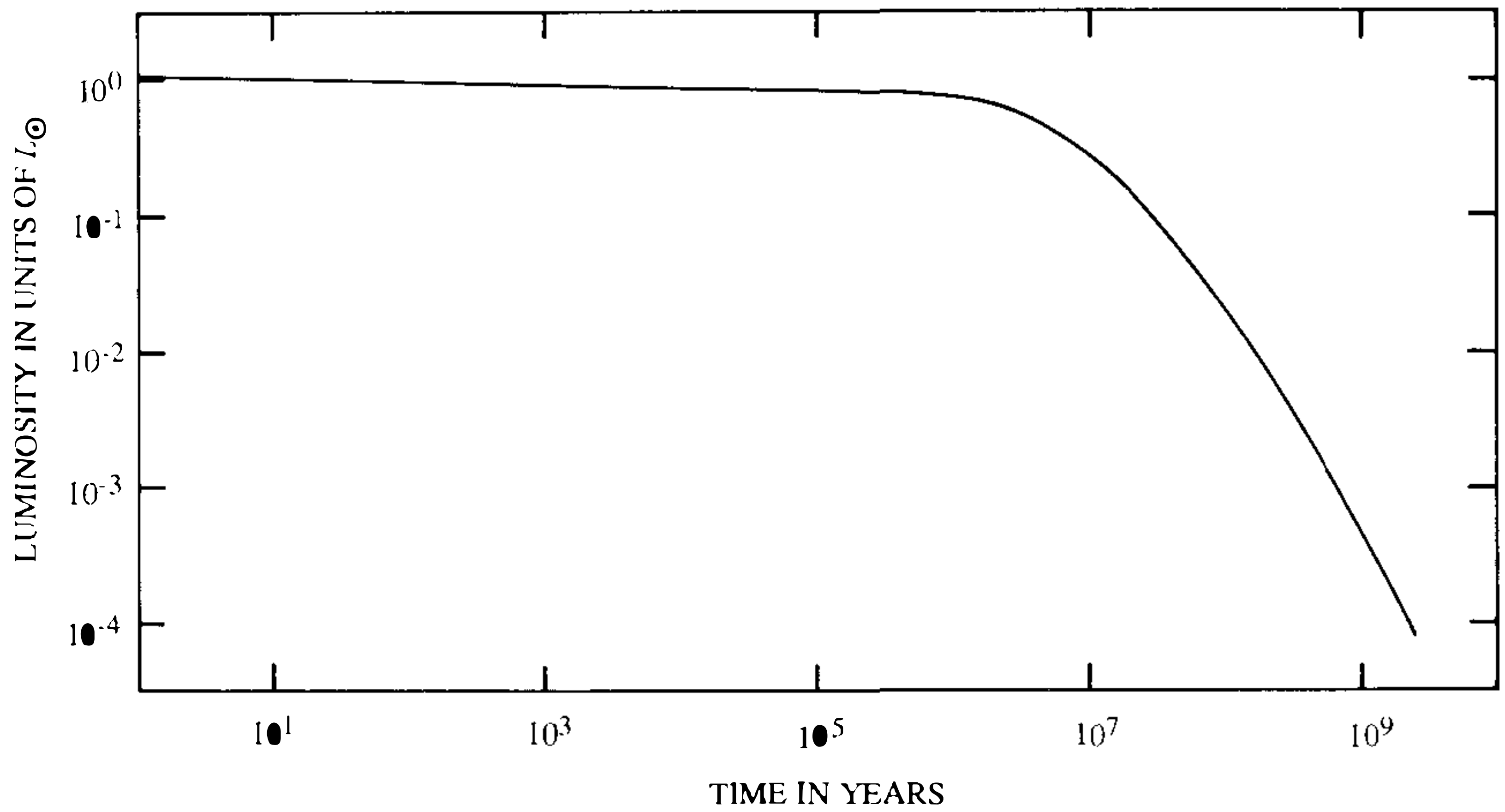
Cooling of White Dwarfs V

Finally, set the cooling rate to the luminosity and obtain

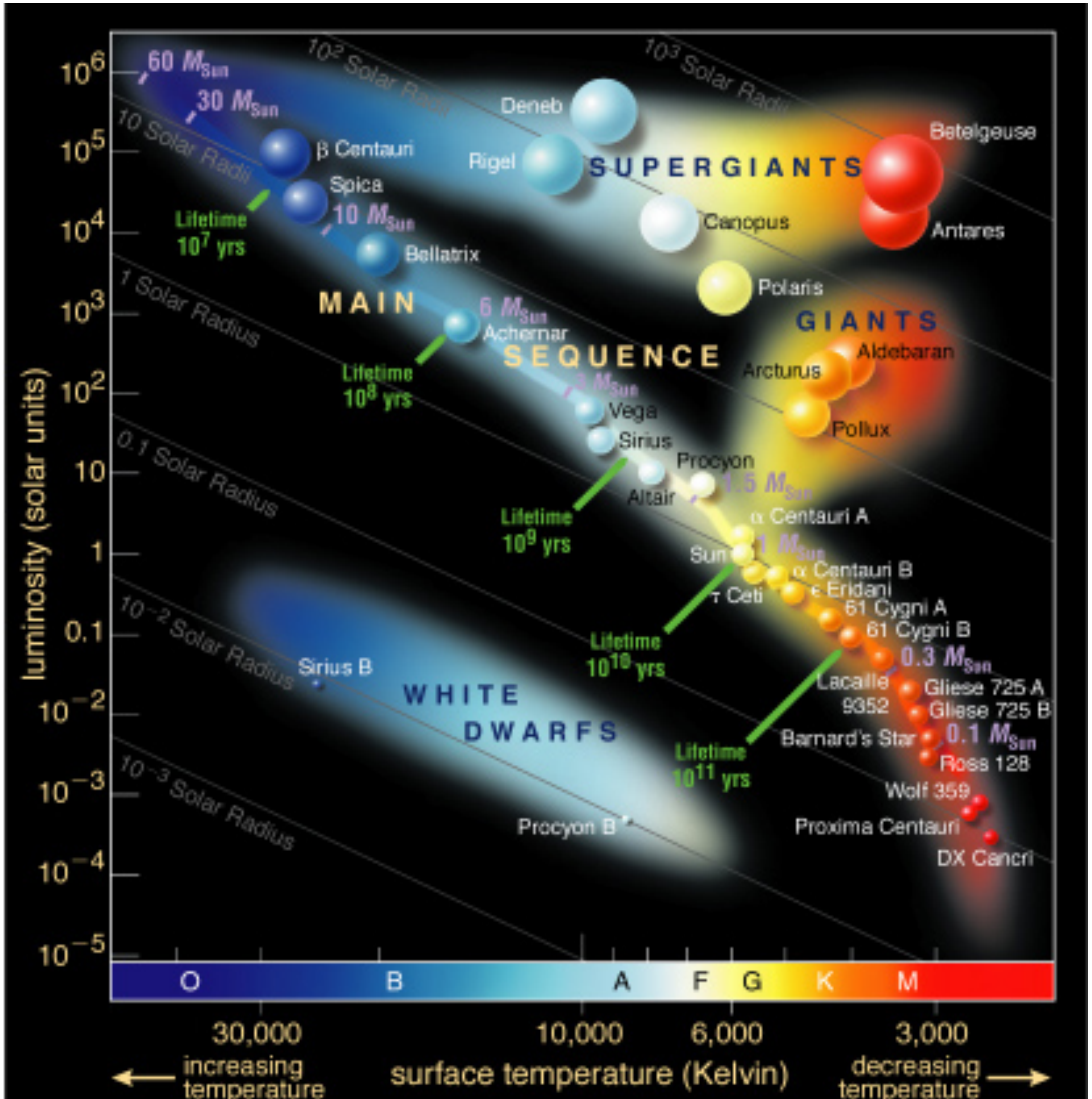
$$\frac{\mathrm{d}E}{\mathrm{d}t} = L \quad \Rightarrow \quad \frac{\mathrm{d}T_I}{\mathrm{d}t} = -\frac{2}{3k}\left[\frac{12m_{\mathrm{H}}}{M_\odot}\right] L_\odot \left[\frac{T_I}{7\times 10^7\ \mathrm{K}}\right]^{7/2}$$

$$\approx 6\ \mathrm{K\ yr^{-1}} \left[\frac{T_I}{7\times 10^7\ \mathrm{K}}\right]^{7/2}$$

The timescale for cooling from a luminosity of about $L_\odot$ to $10^{-4}L_\odot$ is 10^9 yr.



Cooling of White Dwarfs VI



- **WD age-dependent cooling sequence**