Stellar Astrophysics, Fall 2024

Problem Set I

Deadline: 5PM of Thursday, October 3, 2024

extended to 5PM of Friday, October 4

1. **Jeans criterion (10%).** The globular cluster M13 in Hercules contains about 0.5 million stars with an average mass of about half the solar mass. Use the Jeans criterion to check whether this cluster could have been formed in the early university just after the time when the universe was cool enough to form neutral atoms. At that time the density of the universe was $\rho \approx 10^{-30}$ g cm^{-3} and the temperature $T \approx 10^4$ K.

2. **Hydrostatic Equilibrium (35%).** Consider a sphere of mass M and radius R. Calculate the gravitational potential energy of the sphere, the average internal pressure needed for hydrostatic equilibrium, and the pressure as a function of radius, r, by assuming the following density distributions.

   (a) (5%) A uniform density which is independent of the distance from the center.

   (b) (15%) A density which increases towards the center according to
   $$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right).$$

   (c) (15%) A density which increases towards the center with a power-law
   $$\rho(r) = \rho_0 \left(\frac{r}{R}\right)^{-p}.$$

3. **Pressure limits in hydrostatic equilibrium (20%).** Useful bounds can be set on the pressure at a center of a star without detailed stellar structure calculations. Consider a star of mass M and radius R. Let $P(r)$ be the pressure at distance r from the center and $m(r)$ be the mass enclosed by a sphere of radius r.

   (a) (5%) Show that in hydrostatic equilibrium the function
   $$F(r) = P(r) + \frac{Gm(r)^2}{8\pi r^4}$$
   decreases with r.

   (b) (5%) Show that the central pressure satisfies the inequality
   $$P_c > \frac{1}{6}\left[\frac{4\pi}{3}\right]^{1/3} G\langle\rho\rangle^{4/3} M^{2/3},$$

where $\langle\rho\rangle$ is the average density.

(c) (10%) Assuming that the density $\rho(r)$ decreases with r, you can derive a tighter lower bound and, in addition, a useful upper bound for the central pressure. Show that

$$P_c > \frac{1}{2}\left[\frac{4\pi}{3}\right]^{1/3} G\,\langle\rho\rangle^{4/3}\,M^{2/3}.$$

In addition, show that

$$P_c < \frac{1}{2}\left[\frac{4\pi}{3}\right]^{1/3} G\,\rho_c^{4/3}\,M^{2/3},$$

where ρ_c is the central density.