Stellar Astrophysics, Fall 2024

Problem Set II

Deadline: 5PM of Thursday, October 24, 2024

1. **Degenerate electrons (10%).** Consider an ideal gas of degenerate, non-relativistic electrons with a concentration n.

   (a) (5%) Derive an expression for the Fermi energy.

   (b) (5%) Assume now that the gas has a temperature T such that the quantum concentration $n_{\rm QNR}$ is equal to the actual concentration n; quantum effects will be important in such a gas, but the electrons will not be completely degenerate. Find the ratio of kT to the Fermi energy.

2. **The pressure in an ideal degenerate electron gas (25%).** In class, we derive the pressure in an ideal degenerate gas of non-relativistic electrons with mass m and density n to be

$$P_{\rm QNR} = \frac{h^2}{5m}\left[\frac{3}{8\pi}\right]^{2/3} n^{5/3},$$

   while that of predominantly ultra-relativistic electrons to be

$$P_{\rm QUR} = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}.$$

   (a) (15%) Use the relativistic relation between energy and momentum, $\epsilon_p^2 = p^2c^2 + m^2c^4$ and let $x = \dfrac{p}{mc}$. Show that the general expression for the pressure in an ideal degenerate gas is

$$P = P_{\rm QUR}\, I(x_F) = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}\, I(x_F),$$

   where $x_F = \dfrac{p_F}{mc}$ and

$$I(x) = \frac{3}{2x^4}\left\{x(1+x^2)^{1/2}\left[\frac{2x^2}{3} - 1\right] + \ln\left[x + (1+x^2)^{1/2}\right]\right\}.$$

   (b) (10%) Confirm that the general expression in (a) for the pressure P in an ideal degenerate gas, in the appropriate limits, reduces to $P_{\rm QNR}$ and $P_{\rm QUR}$, respectively.