Stellar Astrophysics, Fall 2024

Problem Set VIII

Deadline: 5PM of Thursday, December 12, 2024

1. **Simple Stellar Models (15%).** Consider a star of mass M and radius R in which the pressure gradient is given by

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4\pi}{3}\, G\rho_c^2\, r\, e^{-r^2/a^2},$$

where a is a length parameter and ρ_c is the central density.

   (a) (10%) Derive an expression for the gravitational potential energy U_g of the star.

   (b) (5%) Show that if the length parameter a is small compared with the radius R, the gravitational potential energy is approximately

$$U_g \approx -\frac{1}{3}\frac{R}{a}\frac{GM^2}{R}.$$

   You will need to work out an approximation with numerical value.
   Hint: Recall that

$$M = m(R) \approx \frac{4\pi\rho_c\, a^3\sqrt{6}}{3}.$$

2. **Radiative pressure support (10%).** In Problem Set I, we have learned that the central pressure P_c supporting a star of mass M satisfies the inequality

$$P_c < \left[\frac{\pi}{6}\right]^{1/3} GM^{2/3}\rho_c^{4/3},$$

where ρ_c is the central density.

   (a) (5%) Assume that part of this pressure, denoted by βP_c, is due to an ideal, classical gas of e^-'s and ions with average mass $\overline{m}$, and that the remaining pressure, denoted by $(1-\beta)P_c$, is due to radiation. Show that the above inequality can be used to derive an upper bound for the quantity $\dfrac{1-\beta}{\beta^4}$.

   (b) (5%) Use this bound to set limits on the fraction of the pressure due to radiation at the center of stars of masses 1, 4, and $40M_\odot$.

3. **The Eddington limit (10%).** The upper limit of the main sequence can also be understood through the maximum luminosity that a star can have without blowing

away hydrogen by radiation pressure. Recall the Thomas scattering which characterizes the interaction between photons and free electrons with a cross-section, $\sigma_T = 6.65 \times 10^{-25}$ cm^2. On the stellar surface, electrons are being pushing outwards by the outgoing photon while the H nuclei, with mass $m_{\rm H}$, are attracted by gravitational force of the stellar mass M. When the stellar matters are completely ionized, show that the maximum luminosity, a.k.a. the Eddington limit, that a star can have and still not spontaneously eject H by radiation pressure is

$$L_{\rm Edd} = \frac{4\pi G M c\, m_{\rm H}}{\sigma_T} = 3.2 \times 10^4 L_\odot \left[\frac{M}{M_\odot}\right].$$